Determinants

Definition.

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the square matrix A, where $a_{ij} = (i,j)^{th}$ element of A.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A

is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A)$.

It is denoted by det (A), |A| or Δ

Note :-

For matrix A, |A| is read as determinant of A and not modulus of A.

Determinant of a matrix of order one :-

Let $A = [a]$ be the matrix of order 1, then determinant of A is defined to be equal to a.

$|A|$ or $\det(A)$ or $\Delta = |a| = a$.

Determinant of a matrix of order two

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix of order 2

then the determinant of A is defined as :

$$\det(A) = |A| = \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11}\, a_{22} - a_{21}\, a_{12}$$

Q $A = \begin{vmatrix} 8 & 5 \\ 10 & 1 \end{vmatrix}$

Then $|A| = 8 - 50 = -42$ Ans.

Determinant of a matrix of order 3×3.

Determinant of a matrix of order three can be determined by expressing it in terms of second order determinants. This is known as expansion of a determinant along a row (or a column). There are six ways of expanding a determinant of order 3 corresponding to each of three rows (R_1, R_2 and R_3) and three columns (C_1, C_2 and C_3) giving the same value as shown below:-

Consider the determinant of square matrix

$$A = [a_{ij}]_{3\times3}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along R_1, we have:

$$|A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22} \cdot a_{33} - a_{23} a_{32}) - a_{12}(a_{21} a_{33} - a_{23} \cdot a_{31})$$
$$+ a_{13}(a_{21} a_{32} - a_{22} a_{31}).$$

Q. $|A| = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

Expanding along R_1, we have

$$\Rightarrow (-1)^{1+1} a_{11}(a_{22} \cdot a_{33} - a_{23} a_{32}) - a_{12}(a_{21} a_{33} - a_{23} \cdot a_{31})$$
$$+ a_{13}(a_{21} a_{32} - a_{22} a_{31})$$

$$\Rightarrow 3(0.0 - (5)) - (-1)(0.0 - (-3)) + (-2)(0 - 0)$$

$$\Rightarrow 3 \times -5 + 1 \times 3 + (-2)$$

$$\Rightarrow -15 + 3 - 2$$

$$\Rightarrow -14 \text{ Ans.}$$

Properties of determinant.

i) The value of determinant remain unchanged if its row and column are interchanged.

$$A = \begin{bmatrix} 8 & 3 & 8 \\ 1 & 2 & 8 \\ 5 & 3 & 1 \end{bmatrix}$$

$$= 8\begin{bmatrix} 2 & 8 \\ 3 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 8 \\ 5 & 1 \end{bmatrix} + 8\begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$$

$$= 8(2-24) - 3(1-40) + 8(3-10)$$

$$= 8(-22) - 3(-39) + 8(-7)$$

$$= -176 + 117 - 56$$

$$= -115$$

$$A = \begin{bmatrix} 8 & 3 & 8 \\ 1 & 2 & 8 \\ 5 & 3 & \end{bmatrix}$$

Note: If follow from above property that A is square matrix, then det(A) = det(A') where A' = transpose of A.

ii) If any two rows of a determinant are interchanged then sign of determinant changes.

iii) If any two rows or columns of a determinant ~~changes~~ are same then the value of determinant is zero.

$$|A| = \begin{vmatrix} 8 & 5 & 6 \\ 3 & 1 & 7 \\ 8 & 5 & 6 \end{vmatrix}$$

$$\begin{aligned} |A| &= 8(6-35) - 5(18-56) + 6(15-8) \\ &= 8(-29) - 5(-38) + 6(7) \\ &= -232 + 190 + 42 \\ &= -232 + 232 \\ &= 0 \quad \text{Ans.} \end{aligned}$$

iv) If each element of a row (or column) of determinant is multiplied by a constant 'K' then the value of determinant is multiplied by K.

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$$

$$\begin{aligned} |A| &= 3(32-35) - 3(24-30) + 4(21-24) \\ &= 3(-3) - 3(-6) + 4(-3) \\ &= -9 + 18 - 12 \\ &= -3 \end{aligned}$$

$R_1 \to KR_1$

$$|A| = \begin{vmatrix} 3k & 3k & 4k \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix}$$

$|A| = 3k(32-35) - 3k(24-30) + 4k(21-24)$

$|A| = 3k(-3) - 3k(-6) + 4k(-3)$

$= -9k + 18k - 12k$

$= -3k$ Ans.

28/12/21

Property - 5

If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.

Example:-

$$A = \begin{vmatrix} a_1 + \lambda_1 & a_2 + \lambda_2 & a_3 + \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$A = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Q. Show that $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$

$\Rightarrow$ L.H.S $= \begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$

$\Rightarrow \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$

$\Rightarrow \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + 2 \begin{vmatrix} a & b & c \\ x & y & z \\ x & y & z \end{vmatrix}$

($R_1 = R_2$ in left), ($R_2 = R_3$ in right) (Property-3)

$\Rightarrow 0 + 2 \times 0$

$= 0 + 0 = 0.$

Property - 6.

If, to each element of any row or column of a determinant, the equimultiples of corresponding elements of other row (or column) are added, then value of determinant remain the same.

i.e. the value of determinant remain same if we apply the operation.

$R_i \rightarrow R_i + KR_j$ OR $C_i \rightarrow C_i + KC_j$

Verification

Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

and $\Delta_1 = \begin{vmatrix} a_1 + Kc_1 & a_2 + Kc_2 & a_3 + Kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Where Δ_1 is obtained by the operation

$R_1 \rightarrow R_1 + KR_3$

Here, we have multiplied the elements of the third row (R_3) by a constant K and added them to the corresponding elements of the first row (R_1).

Symbolically :-

⇒ We write this equation as

$R_1 \rightarrow R_1 + KR_3$

Now again

$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} Kc_1 & Kc_2 & Kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + k \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$= \Delta + 0$ (since R_1 and R_3 are same)

Hence $\boxed{\Delta_1 = \Delta}$

Note :-

1) If Δ_1 is the determinant obtained by applying

$$R_i \to kR_i \quad \text{or} \quad C_i \to kC_i$$

to the determinant Δ, then

$$\Delta_1 = k\Delta.$$

ii) If more than one operation like

$R_i \to R_i + kR_j$ is done in one ~~step~~, step, care should be taken to see that a row that is affected in one operation should not be used in another operation. A similar remark applies to column operation

Example:

Prove that.

$$\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6a+3c \end{vmatrix} = a^3$$

→ Solⁿ:

$$L.H.S = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6a+3c \end{vmatrix}$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$,

We have,

$$\begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$$

Now applying $R_3 \to R_3 - 3R_2$, we have

$$= \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 0 & a \end{vmatrix}$$

Expanding along C_1, we have

$$2a \begin{vmatrix} a & 2a+b \\ 0 & a \end{vmatrix} + 0 \begin{vmatrix} a+b & a+b+c \\ 0 & a \end{vmatrix}$$

$$+ 0 \begin{vmatrix} a+b & a+b+c \\ a & 2a+b \end{vmatrix}$$

$$= a(a^2 - 0) - 0 + 0$$

$$= a^3 = R.H.S \quad \text{proved.}$$

☆ Area of TRIANGLE

In earlier class, we have studied that the area of a traingle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by expression.

Then,

$$\text{Area of } \Delta = \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

Now this expression can be written in the form of a determinant as

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Note:

i) Since area is a positive quantity we always take the absolute value of the determinant.

ii) If Area is given, use both positive and negative value of the determinant for calculation.

iii) The area of the triangle formed by three ~~points~~ collinear points is zero.

Example :-

Find the area of the triangle whose vertices are (3, 8), (-4, 2) and (5, 1).

⇒ The area of triangle is given by.

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[3(2-1) + (-4)(1-8) + 5(8-2)\right]$$

$$= \frac{1}{2}\left[3 + 28 + 30\right]$$

$$= \frac{1}{2} \times 61$$

$$= \frac{61}{2}$$

☆ Minors and Cofactors

Definition of Minors :-

Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its ith row and j^{th} column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

Note :

Minor of an element of a determinant of order $n (n \geq 2)$ is a determinant of order $n-1$.

Example :-

Find the minor of element 6 in the determinant.

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Sol:→ Since 6 lies in the second row and third column, its minor M_{23} is given by

$$M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$$

$$= 8 - 14$$

$$= -6$$

Definition of Cofactor.

Cofactor of an element a_{ij}, denoted by A_{ij} is defined by

$\boxed{A_{ij} = (-1)^{i+j} \cdot M_{ij}}$, where M_{ij} is minor of a_{ij}.

Example:

Find minor and cofactors of all the element of the determinant.

$$\begin{vmatrix} 1 & -3 \\ 5 & 6 \end{vmatrix}$$

Sol:- Minor of the element a_{ij} is M_{ij}.

Then,

$a_{11} = 1$, so, $M_{ij} = M_{11} =$ Minor of $a_{11} = 6$.

$M_{12} =$ Minor of element $a_{12} = 5$

$M_{21} = -3$

$M_{22} = 1$

Now, cofactor of a_{ij} is A_{ij} so,

$A_{11} = (-1)^{1+1} \cdot M_{11} = (-1)^2 \cdot 6 = 6$

$A_{12} = (-1)^{1+2} \cdot M_{12} = (-1)^3 \cdot 5 = -5$

$A_{21} = 3$

$A_{22} = 1$.

<u>Note</u> :-

It is denoted by C_{ij} or A_{ij}.

ADJOINT AND INVERSE OF A MATRIX.

Adjoint of a Matrix.

The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$, denoted by Adj A is defined as the transpose of the matrix $[A_{ij}]_{n \times n}$.

where A_{ij} is the cofactor of element a_{ij}.

$$\text{Suppose } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then.}$$

matrix formed by cofactors of each element is

$$C = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

where $A_{11}, A_{12}, A_{13}, \ldots$ are cofactors of elements $a_{11}, a_{12}, a_{13}, \ldots$ respectively.

AND.

$$\text{Adj } A = C^T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$\text{Adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Example :-

Find adj A For $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Sol :-> $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

$A_{11} = 4$, $A_{12} = -1$, $A_{21} = -3$, $A_{22} = 2$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$ Ans

Theorem 1 :-

If A be any given square matrix of order n, then

$$A(\text{adj } A) = (\text{adj } A) A = |A| I$$

where I is the identity matrix

Proof :

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\therefore A(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}.$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= |A|\, I$$

$$\therefore A(\text{adj } A) = (\text{adj } A) A = |A| I.$$

Example :-

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\therefore \text{L.H.S.} = A(\text{adj } A)$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 4 - 3(-1) & -6+6 \\ 4-4 & -3+8 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 5I$$

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 8 - 3 = 5$$

$\therefore$ R.H.S $= |A| I = 5I$ Proved.

☆ Singular Matrix.

A square matrix A whose determinant is zero is called a singular matrix.

Example :-

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$$

$\therefore |A| = 8 - 8 = 0$

then the determinant A is zero.

$\therefore$ A is a singular matrix.

Non Singular Matrix :

A square matrix whose determinant is not zero is called a non-singular matrix.

Example :-

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0.$$

Therefore A is a non-singular matrix.

Theorem 2 :-

If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.

Example :

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 5 & 1 \end{bmatrix}$

$$\therefore AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+10 & 3+2 \\ 0+20 & 9+4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 5 \\ 20 & 13 \end{bmatrix}$$ is a non singular matrix.

AND

BA is a also a non-singular matrix.

Theorem 3 :

The determinant of the product of matrices is equal to product of their respective determinants, that $|AB| = |A||B|$, where A and B are square matrices of the same order.

Example :

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5+2 & 3+4 \\ 15+2 & 9+8 \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ 19 & 17 \end{bmatrix}$$

$$L.H.S = |AB| = \begin{vmatrix} 7 & 7 \\ 19 & 17 \end{vmatrix}$$

$$= 7 \times 17 - 7 \times 15$$
$$= 7(-2)$$
$$= -14$$

$$R.H.S = |A|.|B| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} . \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= (4-6)(10-3) = (-2)(7)$$

$$= -14 = L.H.S$$

Proved.

Theorem 4 :-

A square matrix A is invertible if and only if A is non-singular matrix.

Proof :-

Let A be an invertible matrix of order n and I be the identity matrix of order n.

There exists a square matrix B of order n

$\therefore AB = BA = I$

We know that :

$$A(\text{adj } A) = (\text{adj } A) A = |A| I$$

Dividing by $|A|$, we have

$$A\left(\frac{\text{adj } A}{|A|}\right) = \left(\frac{\text{adj } A}{|A|}\right) = \frac{|A| I}{|A|}$$

$$\text{or, } A\left(\frac{\text{adj } A}{|A|}\right) = \left(\frac{\text{adj } A}{|A|}\right) A = I$$

$$\text{Now, } \left(\frac{\text{adj } A}{|A|}\right) A = I$$

Post multiplication of A^{-1},

$$\therefore \left(\frac{\text{adj } A}{|A|}\right) A . A^{-1} = I A^{-1}$$

$$\text{or, } \left(\frac{\text{adj } A}{|A|}\right) I = A^{-1}$$

$$\boxed{\therefore A^{-1} = \frac{\text{adj } A}{|A|}}$$

Inverse of a matrix :-

Two non-singular matrices A and B are called inverse of each other iff $AB = BA = I$.

Inverse of matrix A is usually denoted by A^{-1}.

It is also called reciprocal of a matrix.

Then by definition, we get.

$$\boxed{\therefore\ AA^{-1} = A^{-1}A = I}$$

Example :-

If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that

$A.\,adj\,A = |A|\,I$, Also find A^{-1}.

Soln: $|A| = 1(16-9) - 3(4-3) + 3(3-4)$

$\Rightarrow 1 \neq 0$.

Now, $A_{11} = 7$, $A_{12} = -1$, $A_{12} = -1$,
$A_{13} = -1$, $A_{21} = -3$, $A_{22} = 1$,
$A_{23} = 0$, $A_{31} = -3$, $A_{32} = 0$, and
$A_{33} = 1$.

$$\therefore \text{adj } A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Now,

$$A(\text{adj } A) = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| . I$$

Also, $A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{1} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$\therefore A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \text{ Ans.}$$

★ **Applications of Determinants And Matrices**

We shall discuss application of Determinants and matrices for solving the system of linear equations in two or three variables and for checking the consistency of the system of linear equations.

~~Contis~~ Consistant System :-

A system of equation is said to be consistant if its solution (one or more) exists.

Inconsistant System.

A system of equations is said to be inconsistent if its solution does not exist.

Solution of system of linear equations using inverse of a matrix.

Let us express the system of linear equations as matrix equations solve them using inverse of the coefficient matrix.

Consider the system of equations.

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

then, the system of equations can be written as.

$$AX = B.$$

$$\text{or, } \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Case 1:-

If A is a nonsingular matrix, then its inverse exists.

Now

$$AX = B \quad \text{——— (1)}$$

Premultiplication of A^{-1} in eq^n (1), we have,

$$A^{-1}AX = A^{-1}B$$

$$\text{or, } IX = A^{-1}B$$

$$\text{or, } \boxed{x = A^{-1}B}$$

Now,

we find A^{-1} then solve the equation

Case II :

If A is a singular matrix then

$$|A| = 0.$$

In this case, we find (adj A) B.

If (adj A) B ≠ 0, then the solution does not exist and the system of equations is called inconsistent.

If (adj A) B = 0, then system may be either consistent or inconsistent according as the system have either infinitely many solutions or no solution.

Example :-

Solve the system of equations.

$$2x + 5y = 1$$
$$3x + 2y = 7.$$

Sol^n:- Given that

$$2x + 5y = 1$$
$$3x + 2y = 7$$

The system of equations can be written as

$$AX = B$$

Where,

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

Now

$$|A| = 4 - 15$$

$$= -11 \neq 0$$

Therefore, A is non-singular matrix and it has unique solution.

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

$$= \frac{1}{|A|} (\text{adj } A)$$

$$= -\frac{1}{11} \begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix}^T$$

$$= \frac{-1}{11} \begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix}^T$$

$$\text{or } A^{-1} = -\frac{1}{11} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

Then

$$X = A^{-1}B$$

or $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{-1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix}$

or, $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{-1}{11}\begin{bmatrix} 2-35 \\ -3+14 \end{bmatrix}$

or, $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{-1}{11}\begin{bmatrix} -33 \\ 11 \end{bmatrix}$

or, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$\therefore \left.\begin{aligned} x &= 3 \\ y &= -1 \end{aligned}\right\}$ Ans.

Cramer's Rule :-

Consider the system linear equations :

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

We find

$$\Delta = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Now we find Δx which is obtained by suppressing the column of coefficient of x and replacing it by the column of constant term d_1, d_2 and d_3 on right hand side.

$$\Delta x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$\Delta z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

$x = \frac{\Delta x}{\Delta}$, $y = \frac{\Delta y}{\Delta}$, $z = \frac{\Delta z}{\Delta}$

where $\Delta \neq 0$.

Q Solve the following by using cramer's rule.

$x - 2y + 3z = 2$
$2x - 3z = 3$
$x + y + z = 6$

$\Rightarrow$

$$\Delta = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 0 & -3 \\ 1 & 1 & 1 \end{vmatrix}$$

$= 1(0+3) + 2(2+3) + 3(2-0)$
$= 3 + 0 + 6$
$= 19$

$$\Delta x = \begin{vmatrix} 2 & -2 & 3 \\ 3 & 0 & -3 \\ 6 & 1 & 1 \end{vmatrix}$$

$= 2(0+3) + 2(3+18) + 3(3-6)$
$= 6 + 42 - 9 = 39.$

$$\Delta y = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & -3 \\ 1 & 6 & 1 \end{vmatrix}$$

Page 54

$= 1(3+18) - 2(2+3) + 3(18-3)$

$= 21 - 10 + 45$

$= 56$

$$\Delta z = \begin{vmatrix} 1 & -2 & 2 \\ 2 & 0 & 3 \\ 1 & 1 & 6 \end{vmatrix}$$

$= 1(0-3) + 2(18-3) + 2(2-0)$

$= -3 + 30 + 4 = 31$

$x = \dfrac{\Delta x}{\Delta} = \dfrac{39}{19}$

$y = \dfrac{\Delta y}{\Delta} = \dfrac{56}{19}$

$z = \dfrac{\Delta z}{\Delta} = \dfrac{31}{19}$